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ABSTRACT

An analytical **functional** can be expressed as a sum of some nonlinear functional expansions which we shall call Fliess's generalized expansions. These nonlinear functional expansions are analogous to Fourier series or integral expansions of response functions of linear systems. The shuffle product which is the characteristic of the noncommutative algebra introduced plays a very significant role in this approach. Moreover what makes this approach more attractive is the possibility of doing all of the noncommutative algebra on a computer in any of the currently available symbolic programming languages such as Macsyma, Reduce, PL1, and Lisp.

Nonlinear functional expansions for the solution of nonlinear ordinary differential equations can be summarized by the newly introduced Laplace-Borel transforms. Some properties of these transforms are obtained by the second author earlier. Some further properties will be given in this paper for the first time.

The main theorem of the paper gives the transform of the response of the nonlinear system as a Cauchy product of its transfer function which is introduced for the first time here and the transform of the input function of the system together with memory effects.

Applications of this new transfer-function approach are given using nonlinear electronic circuits. Two categories of applications are presented, namely,

- analysis of circuits
- synthesis of circuits.

We would like to remind the reader that various other examples can be given from other nonlinear dynamical systems; for example nonlinear aerodynamics, nonlinear flight mechanics in which cases these two classes of problems can be called either **direct** problems or **inverse** problems.

INTRODUCTION

The solution of dynamic problems by classical differential equation analysis is arduous, so that various methods of transform calculus have been developed to ease the burden and increase the

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understanding. It is interesting to note that such **modern** techniques stem from the work of 19th century mathematicians such as Fourier, Laplace, Cauchy, and others.

In this paper we shall develop a methodology to study nonlinear systems via transform methods. In particular we shall use the Laplace Borel transforms which are discussed in references 1, 2, and 3.

The dynamic performance of any initially dead system can be readily described by the frequency response function, $G(jw)$, thus:

$$G(jw) = \frac{\mathcal{F}_o}{\mathcal{F}_i} \tag{1}$$

where $\mathcal{F}$ denotes the Fourier transform. This notion is closely related to the transfer function, $G(s)$, where

$$G(s) = \frac{\mathcal{L}_o}{\mathcal{L}_i} \tag{2}$$

where $\mathcal{L}$ denotes the Laplace transform. The frequency response function and the transfer function are interchangeable by the substitution $s = jw$. Thus the Fourier transform of the system output, $F_0(jw)$ is given by

$$\mathcal{F}_o(jw) = G(jw).\mathcal{F}_i(jw) \tag{3}$$

where $\mathcal{F}_i(jw)$ is the input to the system expressed as a function of frequency either by the Fourier series for periodic functions or by the Fourier Integral for aperiodic functions. The Fourier transform enables a system response to transient excitations to be evaluated in terms of steady-state responses to sinusoidal excitations. Fourier methods have direct application to a few problems which are less easily solved by the Laplace transform:

1. Random problems (i.e., noise and telecommunications) in which the input function can best be expressed as a frequency spectrum (i.e., a Fourier integral).
2. Transformation of functions which are nonzero for negative t and are, therefore, not Laplace transformable.

To find a Fourier pair from a Laplace pair:

1. If the Laplace transform, $F(s)$, has poles on or to the right of the imaginary axis there is no Fourier transform; i.e., $F(s) = \frac{1}{s}$ or $F(s) = \frac{1}{s-a}$ have no Fourier equivalent.
2. Substitute jw for s in $F(s)$ to give $F(jw)$.
3. Note that in using this method $f(t)$ is zero for negative t.

The Laplace-Borel transforms can be summarized as operators which we can obtain from the Laplace transformations as follows:

$$|sF(s)|_{s=x_0^{-1}} \tag{4}$$

except that the algebra on the noncommutative variable x_0 is richer. We have another type of product called **shuffle product** (Le mélange) in addition to Cauchy product. It is the shuffle product which provides the mechanism for us to take care of the nonlinear terms. The shuffle product and some related properties are presented in reference 1. The connection between the Laplace and Fourier transforms is analogous to the one in between the Laplace-Borel and Fourier-Borel transforms. We can generalize the Laplace-Borel transforms to Fourier-Borel transforms in the same way that Fourier transforms are generalized from Laplace transforms.

We have introduced the following section to make the paper self-contained. For some basic development we refer the reader to references 1 and 2.

For an analytical functional or function $f(t)$ we have the following expansion:

$$f(t) = \sum_{n\geq 0} a_n \frac{t^n}{n!} \tag{5}$$

For this expansion, as shown by Fliess and later by Ünal in references 1 and 2 there exists a corresponding generating power series:

$$G = \sum_{n\geq 0} a_n x_0^n \tag{6}$$

in which x_0 is the noncommutative variable. Then the Laplace-Borel transformation is defined as:

$$\mathcal{LB}[f(t)] \equiv G \tag{7}$$

$$G \equiv \sum_{n\geq 0} a_n x_0^n \tag{8}$$

We shall give the following theorem as one of the basic theorems

Theorem 1 : *For an analytic function f(t), there exist a corresponding function $F(x_0)$ of non-commutative variable x_0 defined by an integral transformation:*

$$F(x_0) = x_0^{-1} \int_0^\infty e^{-t/x_0} f(t)dt \tag{9}$$

which is the explicit form of the Laplace-Borel transformation.

Proof: Let us consider an analytic function $f(t)e^{-t/x_0}$ instead of $f(t)$ and integrate from $t = 0$ to $t = \infty$. When we multiplied $f(t)$ with e^{-t/x_0} which is a convergence factor we make $f(t)e^{-t/x_0}$ absolutely integrable even if $f(t)$ is not. The integral I becomes

$$I = \int_0^\infty \sum_{n\geq 0} a_n \frac{t^n}{n!} e^{-t/x_0} dt \tag{10}$$

$$= \sum_{n\geq 0} a_n \int_0^\infty \frac{t^n}{n!} e^{-t/x_0} dt \tag{11}$$

Next we shall use the chain rule with the usual chain rule notation:

$$u = \frac{t^n}{n!} \tag{12}$$

$$dv = e^{-t/x_0} dt \tag{13}$$

Substitute these in I,

$$I = \sum_{n\geq 0} a_n (uv\,|_0^\infty - \int_0^\infty v du) \tag{14}$$

$$= \sum_{n\geq 0} a_n x_0 \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-t/x_0} dt \tag{15}$$

If we continue the integration similarly, we obtain

$$I = x_0 \sum_{n\geq 0} a_n x_0^n \tag{16}$$

$$= x_0 \mathcal{LB}[f(t)] \tag{17}$$

Therefore, we can write

$$\mathcal{LB}[f(t)] \equiv F(x_0) \tag{18}$$

$$= \int_0^\infty x_0^{-1} e^{-t/x_0} f(t) dt \tag{19}$$

Next we shall give some examples to illustrate how this integral transform is obtained.

- Example 1:

  Laplace-Borel transform of the unit step function is as follows :

  Let the unit step function be denoted by $u(t)$ then

$$\mathcal{LB}[u(t)] = \int_0^\infty x_0^{-1} e^{-x_0^{-1} t} dt \tag{20}$$

$$= -x_0^{-1} \frac{e^{-x_0^{-1}}}{x_0^{-1}} |_0^\infty \tag{21}$$

$$= 1 \tag{22}$$

- Example 2:

  Laplace-Borel transform of $f(t) = tu(t)$ is as follows

$$F(x_0) = \mathcal{LB}[tu(t)] \tag{23}$$

$$= \int_0^\infty x_0^{-1} e^{-x_0^{-1} t} dt \tag{24}$$

  Integration by parts with $u = t$ and $dv = e^{-x_0^{-1} t} dt$

$$F(x_0) = \mathcal{LB}[tu(t)] \tag{25}$$

$$= x_0 \tag{26}$$

- Example 3:

  Laplace-Borel transform of $f(t) = t^n u(t)$ is as follows

$$F(x_0) = \int_0^\infty x_0^{-1} t^n e^{-x_0^{-1} t} dt \tag{27}$$

  Applying the integration by parts again like in the previous example we have

$$F(x_0) = \mathcal{LB}[t^n u(t)] \tag{28}$$

$$= n! x_0^n \tag{29}$$

- Example 4:

  Laplace-Borel transform of $f(t) = e^{at}u(t)$ is as follows

$$F(x_0) = \mathcal{LB}[e^{-at}u(t)] \tag{30}$$

$$\int_0^\infty x_0^{-1} e^{-at} e^{-x_0^{-1}t} dt \tag{31}$$

$$= \int_0^\infty x_0^{-1} e^{-(a+x_0)t} dt \tag{32}$$

$$F(x_0) = \mathcal{LB}[e^{-at}u(t)] \tag{33}$$

$$= \frac{1}{1+ax_0} \tag{34}$$

Transform Theorems

Nonlinear differential equations in the time domain are transformed into nonlinear algebraic equations in the transform domain x_0-domain by Laplace-Borel transforms. To establish the relationships between the operations in the two domains, a series of theorems are developed and their applications are illustrated with suitable examples.

Theorem 2 (Linearity Theorem) *: If*

$$\mathcal{LB}[f_1(t)] = F_1(x_0) \tag{35}$$

and

$$\mathcal{LB}[f_2(t)] = F_2(x_0) \tag{36}$$

then

$$\mathcal{LB}[af_1(t) + bf_2(t)] = aF_1(x_0) + bF_2(x_0) \tag{37}$$

where a and b are constants.

This relationship follows directly from the definition of the Laplace-Borel transform. Its principal use is in the decomposition of time functions and transforms to simplify the transformations and inversions. For example, suppose $f(t) = sinw_0t$;

$$\mathcal{LB}[sinw_0t] = \mathcal{LB}[\frac{e^{jw_0t} - e^{-jw_0t}}{2j}] \tag{38}$$

From the earlier example we have

$$\mathcal{LB}[sinw_0t] = 1/2j[\frac{1}{1-jw_0x_0} - \frac{1}{1+jw_0x_0}] \tag{39}$$

$$= \frac{w_0x_0}{1+w_0^2x_0^2} \tag{40}$$

Theorem 3 *: If a causal function $f(t)u(t)$ is delayed along the t-axis by an amount t_0 its Laplace-Borel transform is given by the transform of the undelayed function multiplied by the exponential $e^{-x_0^{-1}t_0}$.*

The proof is as follows :

$$\mathcal{LB}[f(t-t_0)u(t-t_0)] = \int_0^\infty x_0^{-1} f(t-t_0)u(t-t_0)e^{-x_0^{-1}t}dt \tag{41}$$

$$= \int_0^\infty x_0^{-1} f(t-t_0)e^{-x_0^{-1}t}dt \tag{42}$$

where $t_0 \geq 0$. Now, we can make a change of variable $t - t_0 = \tau$ and we obtain

$$\mathcal{LB}[f(t-t_0)u(t-t_0)] = \int_0^\infty x_0^{-1} f(\tau)e^{-x_0^{-1}(t_0+\tau)}d\tau \tag{43}$$

$$= e^{-x_0^{-1}t_0} \int_0^\infty x_0^{-1} f(\tau)e^{-x_0^{-1}\tau}d\tau \tag{44}$$

$$= e^{-x_0^{-1}t_0}\mathcal{LB}[f(\tau)] \tag{45}$$

$$= e^{-x_0^{-1}t_0}F(x_0) \tag{46}$$

As an example of the use of the t_0 shift theorem, we shall determine the Laplace-Borel transform of a rectangular pulse function with a pulse width of T and amplitude of unity. Such a function is expressed as

$$f_p(t) = u(t) - u(t-\tau) \tag{47}$$

The Laplace-Borel transform of $f_p(t)$ is

$$\mathcal{LB}[f_p(t)] = \mathcal{LB}[u(t)] - \mathcal{LB}[u(t-\tau)] \tag{48}$$

$$= 1 - e^{-x_0^{-1}\tau} \tag{49}$$

The pulse function finds a variety of applications in formation of pulse type signals since a valid time function description of any signal extending over the interval t_0 to $t_0 + T$ can be obtained by multiplying the generating signal by $f_p(t - t_0)$.

Theorem 4 (Scale Change Theorem) : *When the independent variable t is multiplied by a constant α (i.e. scaled by α) the corresponding transform is given by*

$$\mathcal{LB}[f(\alpha t)] = F(\alpha x_0) \tag{50}$$

The proof is as follows :

$$\mathcal{LB}[f(\alpha t)] = \int_0^\infty x_0^{-1} f(\alpha t)e^{-x_0^{-1}t}dt \tag{51}$$

Now we make a change of variable namely, $\tau = \alpha t$ and we obtain

$$\mathcal{LB}[f(\alpha t)] = \int_0^\infty (\alpha x_0)^{-1} f(\tau)e^{-(\alpha x_0)^{-1}\tau}d\tau \tag{52}$$

$$= F(\alpha x_0) \tag{53}$$

We can make use of this theorem and the transforms derived for normalized time functions can be modified to cover a wide range of related functions.

Theorem 5 (Differentiation Theorem) : *Laplace-Borel transform of the time derivative of a function $f(t)$ is*

$$\mathcal{LB}[d/dt f(t)] = x_0^{-1}[F(x_0) - f(0)] \tag{54}$$

where $F(x_0)$ is the Laplace-Borel transform of the function $f(t)$.

Proof : We shall use the definition of the Laplace-Borel transform again

$$\mathcal{LB}[d/dt f(t)] = \int_0^\infty x_0^{-1} \frac{df(t)}{dt} e^{-x_0^{-1}t} dt \tag{55}$$

Integration by parts gives ($u = e^{-x_0^{-1}t}$ and $v = f(t)$):

$$\mathcal{LB}[d/dt f(t)] = x_0^{-1}[e^{-x_0^{-1}t} f(t)] |_0^\infty + \int_0^\infty x_0^{-1} f(t) e^{-x_0^{-1}t} dt] \tag{56}$$

since $f(t)e^{-x_0^{-1}t}$ approaches to zero as $t \Longrightarrow \infty$. It follows that

$$\mathcal{LB}[d/dt f(t)] = x_0^{-1}[F(x_0) - f(0)] \tag{57}$$

Hence we establish here that the differentiation in time domain corresponds to multiplication by x_0^{-1} in the x_0-transform domain after a constant which is equal to the value of the function at $t = 0$ is subtracted. We demonstrate the application of this theorem as follows:

$$\mathcal{LB}[cos w_0 t] = \mathcal{LB}[1/w_0 \frac{d}{dt} sin w_0 t] \tag{58}$$

$$= x_0^{-1} 1/w_0 \mathcal{LB}[sin w_0 t - sin(0)] \tag{59}$$

$$\mathcal{LB}[cos w_0 t] = x_0^{-1} \frac{x_0}{1 + w_0^2 x_0^2} \tag{60}$$

$$= \frac{1}{1 + w_0^2 x_0^2} \tag{61}$$

As an extension of this theorem one can easily obtain the expression for the transformation of the second time derivative as

$$\mathcal{LB}[\frac{d^2}{dt^2} f(t)] = x_0^{-1}[x_0^{-1} F(x_0) - x_0^{-1} f(0) - f'(0)] \tag{62}$$

$$= x_0^{-2} F(x_0) - x_0^{-2} f(0) - x_0^{-1} f'(0) \tag{63}$$

Theorem 6 (Integration) : *Laplace-Borel transform of the definite integral*

$$\mathcal{LB}[\int_0^t f(\lambda) d\lambda] \tag{64}$$

is equal to

$$= x_0 F(x_0) \tag{65}$$

Proof :

$$\mathcal{LB}[\int_0^t f(\lambda)d\lambda] = \int_0^\infty x_0^{-1}[\int_0^t f(\lambda)d\lambda]e^{-x_0^{-1}t}dt \tag{66}$$

Integration by parts with $u = \int^t f(\lambda)d\lambda$ and $dv = e^{-x_0^{-1}t}dt$

$$\mathcal{LB}[\int_0^t f(\lambda)d\lambda] = -\frac{x_0^{-1}}{x_0^{-1}}e^{-x_0^{-1}t}\int_0^t f(\lambda)d\lambda\,|_0^\infty + \frac{1}{x_0^{-1}}\int_0^\infty x_0^{-1}e^{-x_0^{-1}t}f(t)dt \tag{67}$$

since $e^{-x_0^{-1}t} \Longrightarrow 0$ for $t \Longrightarrow \infty$ and $\int_0^t f(\lambda)d\lambda\,|_{t=0} = 0$ then

$$\mathcal{LB}[\int_0^t f(\lambda)d\lambda] = x_0\int_0^\infty x_0^{-1}e^{-x_0^{-1}t}f(t)dt \tag{68}$$

$$\mathcal{LB}[\int_0^t f(\lambda)d\lambda] = x_0F(x_0) \tag{69}$$

which completes the proof.

Theorem 7 (Convolution) *: Let $f_1(t), F_1(x_0)$ and $f_2(t), F_2(x_0)$ be two Laplace-Borel transform pairs, then the Laplace-Borel transform of the convolution of the $f_1(t)$ and $f_2(t)$ is given by*

$$\mathcal{LB}[f_1(t) * f_2(t)] = x_0F_1(x_0)F_2(x_0) \tag{70}$$

Proof : Let us try to find the inverse transform of the product

$$x_0F_1(x_0)F_2(x_0) = x_0F_1(x_0)[\int_0^\infty x_0^{-1}f_2(t)e^{-x_0^{-1}t}dt] \tag{71}$$

Changing the variable of integration and bringing $F_1(x_0)$ under the integral sign gives

$$x_0F_1(x_0)F_2(x_0) = \int_0^\infty F_1(x_0)f_2(\lambda)e^{-x_0^{-1}\lambda}d\lambda \tag{72}$$

Noticing that

$$F_1(x_0)e^{-x_0^{-1}\lambda} = \mathcal{LB}[f_1(t-\lambda)u(t-\lambda)] \tag{73}$$

we can write

$$x_0F_1(x_0)F_2(x_0) = \int_0^\infty [\int_0^\infty x_0^{-1}f_1(t-\lambda)u(t-\lambda)e^{-x_0^{-1}t}dt]f_2(\lambda)d\lambda \tag{74}$$

$$= \int_0^\infty x_0^{-1}[\int_0^\infty f_1(t-\lambda)u(t-\lambda)f_2(\lambda)d\lambda]e^{-x_0^{-1}t}dt \tag{75}$$

Noting that λ can not exceed t because of $u(t-\lambda)$, we can write

$$x_0F_1(x_0)F_2(x_0) = \int_0^\infty x_0^{-1}[\int_0^t f_1(t-\lambda)f_2(\lambda)d\lambda]e^{-x_0^{-1}t}dt \tag{76}$$

then we have

$$x_0F_1(x_0)F_2(x_0) = \mathcal{LB}[\int_0^t f_1(t-\lambda)f_2(\lambda)d\lambda] \tag{77}$$

or using the shorthand notation for convolution,

$$\mathcal{LB}[f_1(t) * f_2(t)] = x_0 F_1(x_0) F_2(x_0) \tag{78}$$

FOURIER-BOREL TRANSFORMS

An analytic function $f(t)$ can be represented as a linear combination of a set of elementary time functions called **basis functions** $\Phi_n(t)$ as shown below:

$$f(t) = \sum_{n=0}^{N} a_n \Phi_n(t) \tag{79}$$

Let us define a basis function $\Phi_n(t)$ and its complex conjugate $\Phi_n^*(t)$ as

$$\Phi_n(t) = jn\beta_0 e^{jn\beta_0 t} \tag{80}$$

$$\Phi_n^*(t) = -jn\beta_0 e^{-jn\beta_0 t} \tag{81}$$

where $n = 0, \mp 1, \mp 2, \ldots, \mp\infty$ and $\beta_0 = \frac{2\pi}{T}$. We can show that these functions are orthogonal so that

$$\frac{1}{T}\int_{t_1}^{t_1+T} \Phi_n(t)\Phi_n^*(t)dt = \lambda_n \tag{82}$$

$$\lambda_n = (n\beta_0)^2 \tag{83}$$

This ensures the desired property for a set of basis functions; namely, the finality of coefficients which allows one to determine any given coefficient without the need for knowing any other coefficient. To determine the coefficients a_n, we multiply both sides of the equation defining $f(t)$ by $\Phi_n^*(t)$ and integrate over the specified interval. This gives

$$\int_{t_1}^{t_1+T} \Phi_n^*(t) f(t) dt = \int_{t_1}^{t_1+T} \Phi_n^*(t)[\sum_{n=0}^{N} a_n \Phi_n(t)]dt \tag{84}$$

$$= \sum_{n=0}^{N} a_n \int_{t_1}^{t_1+T} \Phi_n^*(t)\Phi_n dt \tag{85}$$

From orthogonality condition we have

$$a_n = \frac{1}{\lambda_n T}\int_{t_1}^{t_1+T} \Phi_n^*(t) f(t) dt \tag{86}$$

Substituting the last equation into the first equation gives

$$f(t) = \sum_{n=-\infty}^{\infty} jn\beta_0 e^{jn\beta_0 t}[\frac{1}{(n\beta_0)^2 T}\int_{-T/2}^{T/2} \frac{n\beta_0}{j} e^{-jn\beta_0 t} f(t) dt] \tag{87}$$

$$= \sum_{-\infty}^{\infty} jn\beta_0 e^{jn\beta_0 t}[\frac{1}{2\pi}\int_{-T/2}^{T/2} \frac{1}{jn\beta_0} e^{-jn\beta_0 t} f(t) dt] \tag{88}$$

Note that $\beta_0 = \frac{2\pi}{T}$ is the lowest frequency component and also the spacing between the harmonics. Now if we let $T \longrightarrow \infty$, the spacing between harmonics will become a differential, that is $\beta_0 \longrightarrow d\beta$. The number of components becomes infinite hence $n \longrightarrow \infty$. The angular frequency of any particular component is given by $n\beta_0$; and the summation formally passes into an integral. If we rewrite the last equation as

$$f(t) = \int_{-\infty}^{\infty} j\beta e^{j\beta t}[\frac{d\beta}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\beta}e^{-j\beta t}f(t)dt] \tag{89}$$

or

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\beta}e^{j\beta}t\{j\beta\int_{-\infty}^{\infty}e^{-j\beta t}f(t)dt\}d\beta \tag{90}$$

It is clear that the inner integral is only a function of the angular frequency since the time is integrated out. Now we shall call the inner integral Fourier-Borel transform of $f(t)$ and we shall denote it as follows:

$$\mathcal{FB}[f(t)] = F(j\beta) \tag{91}$$

$$F(j\beta) = j\beta\int_{-\infty}^{\infty}e^{-j\beta t}f(t)dt \tag{92}$$

Similarly we shall define the inverse Fourier-Borel transform as

$$\mathcal{FB}^{-1}[F(j\beta)] = f(t) \tag{93}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(j\beta)^{-1}F(j\beta)e^{j\beta t}d\beta \tag{94}$$

The functions $f(t)$ and $F(j\beta)$ are called the Fourier-Borel transform pairs. It should be noted that if we choose the basis functions and their complex conjugates as

$$\Phi_n(t) = (\alpha + jn\beta_0)e^{jn\beta_0} \tag{95}$$

$$\Phi_n^*(t) = (\alpha - jn\beta_0)e^{-jn\beta_0} \tag{96}$$

In this case the Fourier-Borel transform pair becomes

$$\mathcal{FB}[f(t)] = F(j\beta) \tag{97}$$

$$F(j\beta) = (\alpha + j\beta)\int_{-\infty}^{\infty}f(t)e^{-j\beta t}dt \tag{98}$$

$$\mathcal{FB}^{-1}[F(j\beta)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\alpha + j\beta)^{-1}f(t)e^{j\beta t}d\beta \tag{99}$$

Let us consider the Fourier-Borel transform of the function $f(t)e^{-\alpha t}$ rather than the function $f(t)$. In this case the factor $e^{-\alpha t}$ is a convergence factor that tends to make $e^{\alpha t}f(t)$ absolutely integrable even if $f(t)$ is not. The Fourier-Borel transform of $f(t)e^{-\alpha t}$ is :

$$\mathcal{FB}[e^{-\alpha t}f(t)] = \int_{-\infty}^{\infty}(\alpha + j\beta)e^{-\alpha t}f(t)e^{-j\beta t}dt \tag{100}$$

$$= \int_{-\infty}^{\infty}(\alpha + j\beta)e^{-(\alpha + j\beta)t}f(t)dt \tag{101}$$

$$= F(\alpha + j\beta) \tag{102}$$

The corresponding inverse transform is

$$\mathcal{F}\mathcal{B}^{-1}[F(\alpha + j\beta)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\alpha + j\beta)^{-1}F(\alpha + j\beta)e^{j\beta t}d\beta \tag{103}$$

the convergence factor can be taken to the right-hand side to give

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\alpha + j\beta)^{-1}F(\alpha + j\beta)e^{(\alpha+j\beta)t}d\beta \tag{104}$$

Next we shall define a new variable $z_0 = \alpha + j\beta$ then we have $dz_0 = jd\beta$ if α is constant. The inverse transform becomes

$$f(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} z_0^{-1}F(z_0)e^{z_0 t}dz_0 \tag{105}$$

Important Properties of Fourier-Borel Transforms

We shall give the following most important properties of the Fourier-Borel transforms without proving them, but interested reader can verify them by using the basic definitions given in this text:

$$[F_1(x_0) + F_2(x_0)]\coprod^{2} = F_1(x_0)\coprod^{2} + 2[F_1(x_0)\coprod F_2(x_0)] + F_2(x_0)\coprod^{2} \tag{106}$$

$$F_1(x_0)\coprod F_2(x_0) = 1/4\{[F_1(x_0) + F_2(x_0)]\coprod^{2} - [F_1(x_0) - F_2(x_0)]\coprod^{2}\} \tag{107}$$

$$\frac{d}{dx_0}[F_1(x_0)\coprod F_2(x_0)] = F_1(x_0)\coprod \frac{d}{dx_0}F_2(x_0) + F_2(x_0)\coprod \frac{d}{x_0}F_1(x_0) \tag{108}$$

$$F_1(x_0)\coprod[F_2(x_0) + F_3(x_0)] = F_1(x_0)\coprod F_2(x_0) + F_1(x_0)\coprod F_3(x_0) \tag{109}$$

$$\alpha[F_1(x_0)\coprod F_2(x_0)] = \alpha F_1(x_0)\coprod F_2(x_0) \tag{110}$$

$$= F_1(x_0)\coprod \alpha F_2(x_0) \tag{111}$$

$$F_1.(F_2\coprod F_3) \neq F_1.F_2\coprod F_3 \tag{112}$$

$$Ln[F_1(x_0)\coprod F_2(x_0)] = Ln[F_1(x_0)] + Ln[F_2(x_0)] \tag{113}$$

$$Ln[F(x_0)\coprod^{n}] = nLn[F(x_0)] \tag{114}$$

$$EX[F_1(x_0)\coprod F_2(x_0)] = EX[F_1(x_0) + F_2(x_0)] \tag{115}$$

$$Ln\{EX[F(x_0)]\} = EX\{Ln[F(x_0)]\} \tag{116}$$

$$= F(x_0) \tag{117}$$

Another very useful result is as follows:

$$\frac{1}{(1 + ax_0)^{(1-n)}}\coprod\frac{1}{(1 - ax_0)} = \frac{1}{(1 - ax_0)^n} \tag{118}$$

We shall prove this result for $n = 2$ and then by the method of mathematical induction more general case can be obtained. For $n = 2$ we have

$$\frac{1}{(1 - ax_0)^2} = (1 + ax_0)\coprod\frac{1}{(1 - ax_0)} \tag{119}$$

Theorem 8 (Shifting) : *The Fourier-Borel transform of a function is related to that of the function multiplied by an exponential function* $e^{-\beta t}$ *as follows*

$$\mathcal{FB}[f(t)e^{-\beta t}] = \frac{z_0}{z_0+\beta}F(z_0+\beta) \tag{120}$$

$$= \frac{z_0}{z_0+\beta}\coprod F(z_0) \tag{121}$$

Proof:

$$\mathcal{FB}[f(t)e^{-\beta t}] = \int_0^\infty z_0 e^{-z_0 t} f(t)e^{-\beta t}dt \tag{122}$$

$$= z_0 \int_0^\infty e^{-(z_0+\beta)t} f(t)dt \tag{123}$$

$$\frac{z_0}{z_0+\beta}\int_0^\infty (z_0+\beta)e^{-(z_0+\beta)t} f(t)dt \tag{124}$$

$$= \frac{z_0}{z_0+\beta}F(z_0+\beta) \tag{125}$$

From Ünal in reference 1 we also have

$$\mathcal{FB}[e^{-\beta t}f(t)] = \mathcal{FB}[e^{-\beta t}]\coprod \mathcal{FB}[f(t)] \tag{126}$$

$$= \frac{z_0}{z_0+\beta}\coprod F(z_0) \tag{127}$$

which completes the proof.

Connection Between Fourier-Borel and Laplace-Borel Transforms

We have the following integral representation for the two-sided or bilateral Laplace-Borel transforms

$$F(z_0) = \int_{-\infty}^\infty z_0 e^{-z_0 t} f(t)dt \tag{128}$$

Let I represent the following integral

$$I = \int_0^\infty f(t)e^{-z_0 t}dt \tag{129}$$

$$= \int_0^\infty \sum_{n\geq 0} a_n \frac{t^n}{n!} e^{-z_0 t}dt \tag{130}$$

$$= \sum_{n\geq 0} a_n \int_0^\infty \frac{t^n}{n!} e^{-z_0 t}dt \tag{131}$$

Let

$$u = \frac{t^n}{n!} \tag{132}$$

$$e^{-z_0 t}dt = dv \tag{133}$$

$$I = \sum_{n\geq 0} a_n [fract^n n! z_0^{-1} e^{-z_0 t}\,|_0^\infty + \int_0^\infty z_0^{-1}\frac{t^{(n-1)}}{(n-1)!}e^{-z_0 t}]dt \tag{134}$$

$$= \sum_{n \geq 0} a_n z_0^{-1} \int_0^{\infty} \frac{t^{(n-1)}}{(n-1)!} e^{-z_0 t} dt \tag{135}$$

$$I = z_0^{1} \sum_{n \geq 0} a_n z_0^{-n} \tag{136}$$

$$z_0 = \frac{1}{x_0} \tag{137}$$

Hence the integral I becomes

$$I = x_0 \sum a_n x_0^n \tag{138}$$

$$= x_0 \mathcal{LB}[f(t)] \tag{139}$$

$$\mathcal{LB}[f(t)] = z_0 I \tag{140}$$

$$\mathcal{LB}[f(t)] = \int_0^{\infty} z_0 f(t) e^{-z_0 t} dt \tag{141}$$

$$= F(z_0) \tag{142}$$

where $z_0 = x_0^{-1}$ in which x_0 is the noncommutative variable.

TRANSFER FUNCTIONS FOR NONLINEAR SYSTEMS

Consider the following class of nonlinear systems with polynomial nonlinearity described by

$$\sum_{i=1}^{n} a_i \frac{d^i}{dt^i} x(t) + \sum_{j=1}^{m} b_j x^j(t) = f(t) \tag{143}$$

We define the operator $\coprod$" as the shuffle product which is defined by Ünal in reference 1 repeated n times and the transfer function is the transform of the response caused by a unit step function with zero initial conditions

$$\mathcal{G}(z_0, \overset{i}{\coprod}) \equiv X(z_0) \mid_{f(t)=u(t)} \tag{144}$$

In the Laplace-Borel transform domain the following nonlinear differential equation becomes

$$\frac{dx(t)}{dt} + k_1 x(t) + k_2 x^2(t) = f(t) \tag{145}$$

$$z_0 [X(z_0) - x(0)] + k_1 X(z_0) + k_2 \left[X(z_0) \coprod X(z_0) \right] = \mathcal{LB}\{f(t)\} \tag{146}$$

From Ünal (ref. 1) we have

$$\mathcal{LB}[u(t)] = 1 \tag{147}$$

with the zero initial conditions (i.e., $x(0) = 0$) the transfer function for this nonlinear differential equation becomes

$$(z_0 + k_1 + k_2 \overset{2-1}{\coprod}) G(z_0, \overset{2-1}{\coprod}) = 1 \tag{148}$$

or

$$G(z_0, \overset{2-1}{\coprod}) = \frac{1}{z_0 + k_1 + k_2 \coprod^{2-1}} \tag{149}$$

Theorem 9 (Main Theorem) *: The Laplace-Borel transform of the response of the nonlinear system considered is equal to the Cauchy product of the transfer function[$G(z_0, \coprod)$] with the Laplace-Borel transform of the function which consists of the forcing function and the initial conditions of the response and all of its higher order derivatives.*

$$X(z_0) = G(z_0, \coprod).\mathcal{LB}\{f(t) + \sum_{k=0}^{i-1} \alpha_i \frac{d^k}{dt^k} x(0) \frac{d^{i-k-1}}{dt^{i-k-1}} \delta(t)\} \tag{150}$$

$$= G(z_0, \coprod).\mathcal{LB}\{f(t) + \sum_{k=0}^{i-1} \alpha_i \frac{d^k}{dt^k} x(0) \frac{d^{i-k}}{dt^{i-k}} u(t)\} \tag{151}$$

Proof:

Let us consider a nonlinear dynamical system described by an n^{th} order nonlinear differential equation with m^{th} order polynomial nonlinearity as follows :

$$\sum_{0}^{n} \alpha_i \frac{d^i}{dt^i} x(t) + k_1 x(t) + \sum_{j=2}^{m} k_j x^j(t) = f(t) \tag{152}$$

We shall demonstrate the proof on the sample problem and then consider the general form.

If the dynamical system has an evolution equation of the first order with quadratic nonlinearity; i.e.,

$$\frac{dx}{dt} + k_1 x(t) + k_2 x^2(t) = f(t) \tag{153}$$

we want to express the Fourier(or Laplace)-Borel transform of the system in terms of its transfer function and the transform of the input function. To do this we shall take the Laplace-Borel transform of the given equation, and hence, we have

$$z_0 X(z_0) - x(0) + k_1 X(z_0) + k_2 X(z_0) \coprod X(z_0) = F(z_0) \tag{154}$$

We defined the transfer function as the Laplace-Borel transform of the output of the system for a unit step function input with zero initial conditions (assuming that the system is initially dead).

$$f(t) = u(t) \tag{155}$$

and

$$x(0) = 0 \tag{156}$$

hence

$$z_0 X(z_0) + k_1 X(z_0) + k_2 X(z_0) \overset{2-1}{\coprod} = 1 \tag{157}$$

or

$$X(z_0) = \frac{1}{z_0 + k_1 + k_2 \coprod} \tag{158}$$

$$= G(z_0, \coprod) \tag{159}$$

Now we go back to the original equation and take the Laplace-Borel transform of it as

$$z_0 X(z_0) + k_1 X(z_0) + k_2 X(z_0) \coprod X(z_0) = F(z_0) + x(0) \tag{160}$$

$$\left(z_0 + k_1 + k_2 \coprod\right) X(z_0) = [F(z_0) + x(0)] \tag{161}$$

or in terms of the transfer function

$$X(z_0) = G(z_0, \coprod)[F(z_0) + x(0)] \tag{162}$$

Notice that for the system of order one the memory effect consists of only the value of the response at the start.

We shall repeat this procedure for the more general case of a dynamical system described by an n^{th} order nonlinear ordinary differential equation with m^{th} order nonlinearity in it:

$$\sum_{i=1}^{n} \alpha_i \frac{d^i}{dt^i} x(t) + k_1 x(t) + \sum_{j=2}^{m} k_j x^j(t) = f(t) \tag{163}$$

$$\sum_{i=1}^{n} z_0^i \alpha_i X(z_0) - \sum_{i=1}^{n} \alpha_i z_0^{i-1} \frac{d^{i-1}}{dx^{i-1}} x(0) + k_1 X(z_0) + \sum_{j=2}^{m} k_j X(z_0) \coprod^{j-1} = F(z_0) \tag{164}$$

or

$$X(z_0)\left(\sum_{i=1}^{n} \alpha_i z_0^i + k_1 + \sum_{j=2}^{m} k_j \coprod^{j-1}\right) = \left[F(z_0) + \sum_{i=1}^{n} \alpha_i z_0^{i-1} \frac{d^{i-1}}{dx^{i-1}} x(0)\right] \tag{165}$$

or

$$X(z_0) = \frac{1}{\left(\sum_{i=1}^{n} \alpha_i z_0^i + k_1 + \sum_{j=2}^{m} k_j \coprod^{j-1}\right)} \left[F(z_0) + \sum_{i=1}^{n} \alpha_i z_0^{i-1} \frac{d^{i-1}}{dx^{i-1}} x(0)\right] \tag{166}$$

or in terms of the transfer function of the system we have:

$$X(z_0) = G(z_0, \coprod^{j-1}). \left[F(z_0) + \sum_{i=1}^{n} \alpha_i z_0^{i-1} \frac{d^{i-1}}{dx^{i-1}} x(0)\right] \tag{167}$$

Notice that for $i = 1$ and $j = 2$ and $\alpha_1 = 1$ the preceding general relation reduces to

$$X(z_0) = \left(z_0 + k_1 + k_2 \coprod\right) [F(z_0) + x(0)] \tag{168}$$

$$= G(z_0, \coprod^{2-1}) [F(z_0) + x(0)] \tag{169}$$

Corollary 1 (Main Corollary) : *For a memoryless system the Laplace-Borel transform of the output of the system is given by the Cauchy product of the transfer function with the Laplace-Borel transform of the input of the system.*

Proof of Corollary: The memory effects are lumped into the second term of the second factor of the Cauchy product; i.e.,

$$\sum_{i=1}^{n} \alpha_i z_0^{i-1} \frac{d^{i-1}}{dx^{i-1}} x(0) \tag{170}$$

which means function itself and its higher order derivatives evaluated at the zero time.

When the system has no memory we shall be left with

$$X(z_0) = G(z_0, \coprod) F(z_0) \tag{171}$$

In other words we can say that the main theorem has the same form as the linear systems if the nonlinear system has no memory.

APPLICATIONS

We shall give examples both from synthesis and analysis of nonlinear circuits as applications to Fourier-Borel based transfer function approach to nonlinear systems.

Representation of Nonlinear Resistance

$$v \equiv R(i)i \tag{172}$$

$$R(i) = R_0 + R_1 i + R_2 i^2 + \dots \tag{173}$$

$$= \sum_{j=0}^{n} R_j i^j \tag{174}$$

$$v = \left(\sum_{j=0}^{n} R_j i^j\right) i \tag{175}$$

$$= \sum_{j=0}^{n} R_j i^{j+1}. \tag{176}$$

By Laplace-Borel transformation

$$V(z_0) = \sum_{0}^{n} R_j I(z_0) \coprod^{j+1} \tag{177}$$

From the definition of system function

$$V(z_0) = R(z_0, \coprod^{j+1}).I(z_0) \tag{178}$$

$$R(z_0, \coprod^{j+1}) = \sum_{j=0}^{n} R_j \coprod \tag{179}$$

and the admittance function follows

$$G(z_0, \coprod^{j+1}) = \frac{1}{\sum_{j=0}^{n} R_j \coprod} \tag{180}$$

Representation of Nonlinear Capacitance

We shall define the impedance and admittance functions in Laplace-Borel domain as:

$$C(v) = \sum_{j=0}^{n} C_j v^j \tag{181}$$

$$Q = C(v)v \tag{182}$$

$$= \sum_{j=0}^{n} C_j v^{j+1} \tag{183}$$

$$i = \frac{d}{dt} Q \tag{184}$$

$$I(z_0) = z_0 Q(z_0) \tag{185}$$

$$I(z_0) = z_0 \sum_{j=0}^{n} C_j v(z_0) \amalg^{(j+1)} \tag{186}$$

$$Y(z_0) = z_0 \sum_{j=0}^{n} C_j \overset{(j+1)}{\amalg} \tag{187}$$

$$Z(z_0) = Y(z_0)^{-1} \tag{188}$$

$$= \frac{1}{z_0} \sum_{j=0} C_j \overset{(j+1)}{\amalg} \tag{189}$$

Representation of Nonlinear Inductance

We shall define the impedance and admittance functions in the Laplace-Borel transform domain as :

$$L(i) = \sum_{j=0}^{n} L_j i^j(t) \tag{190}$$

$$\Phi(t) = L(i)i \tag{191}$$

$$= \sum_{j=0}^{n} L_j i^{j+1} \tag{192}$$

$$\frac{d\Phi}{dt} = v \tag{193}$$

$$= \frac{d}{dt} \sum_{j=0}^{n} L_j i^{j+1} \tag{194}$$

the derivative theorem applied once yields

$$v(z_0) = z_0 \Phi(z_0) \tag{195}$$

$$= z_0 \sum_{j=0}^{n} L_j I(z_0) \overset{j+1}{\amalg} . \tag{196}$$

We can write using the operator symbolism whereby $\amalg$ to any power indicates the shuffle product which is discussed by Ünal in reference 1.

$$v(z_0) = Z(z_0) I(z_0) \tag{197}$$

$$Z(z_0) = z_0 \sum_{j=0}^{n} L_j \coprod^{j+1} \tag{198}$$

$$Y(z_0) = Z^{-1}(z_0) \tag{199}$$

$$= \frac{1}{z_0 \sum_{j=0} L_j \coprod^{j+1}} \tag{200}$$

A Nonlinear Circuit

Figure 1 shows the first nonlinear system which we shall characterize by the following nonlinear ordinary first-order differential equation.

$$i(t) = i_R + i_C + i_{RN} \tag{201}$$

$$= v/R + C\frac{dv(t)}{dt} + i_{RN} \tag{202}$$

$$C\frac{dv}{dt} + \frac{1}{R}v + k_2v^2 = i(t) \tag{203}$$

$$\frac{dv}{dt} + k_1v + k_2v^2 = i(t) \tag{204}$$

$$z_0[V(z_0 - v(0)] + k_1V(z_0) + k_2V(z_0)\coprod^{2-1} = I(z_0) \tag{205}$$

which becomes

$$z_0V(z_0) + k_1V(z_0) + k_2V(z_0)\coprod^{2-1} = I(z_0) \tag{206}$$

The system's transfer function is

$$G(z_0, \coprod^{2-1}) = \frac{1}{z_0 + k_1 + k_2\coprod^{2-1}} \tag{207}$$

We notice that in this example the system's transfer function is an impedance; i.e., $G \Longrightarrow Z$.

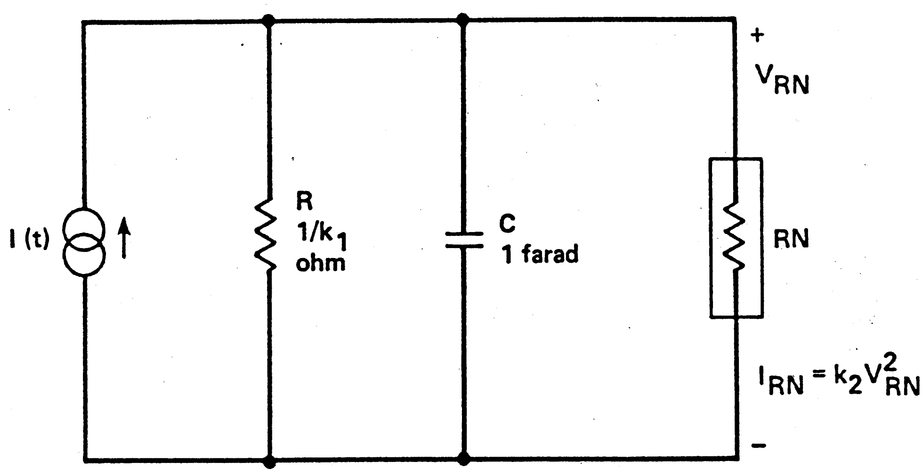

Figure 1.- A nonlinear circuit.

A Synthesis Problem

Let us suppose that we are given an impedance function $Z(z_0)$ as

$$Z(z_0) = \frac{1}{z_0 + k_1 + k_2 \coprod^{2-1}} \tag{208}$$

$$Y(z_0) = \frac{1}{Z(z_0)} \tag{209}$$

$$= z_0 + k_1 + k_2 \coprod^{2-1} \tag{210}$$

In figure 2 we shall denote the three parallel elements by

$$Y_1 = z_0 \tag{211}$$

$$Y_2 = k_1 \tag{212}$$

$$Y_3 = k_2 \coprod^{2-1} \tag{213}$$

$$Y_1(z_0) = z_0 \tag{214}$$

$$\Longrightarrow C = 1 \tag{215}$$

$$Y_2(z_0) = k_1 \tag{216}$$

$$\Longrightarrow R = \frac{1}{k_1} \tag{217}$$

$$Y_3(z_0) = k_2 \coprod^{2-1} \tag{218}$$

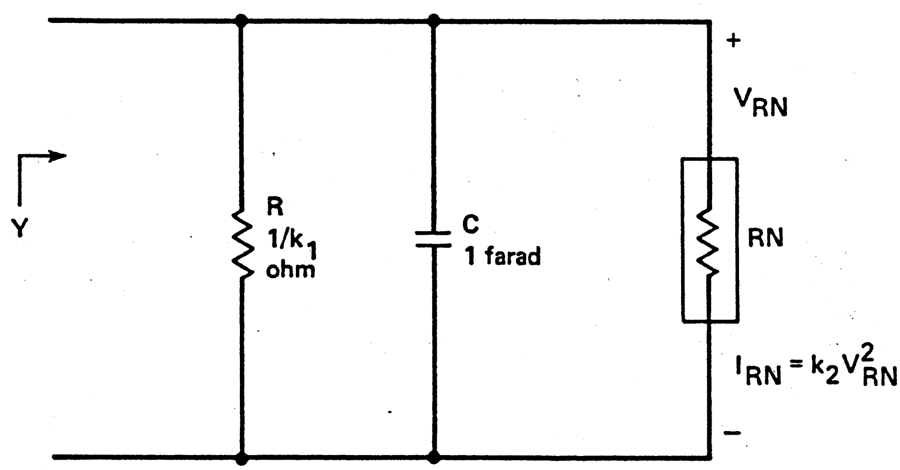

Figure 2.- A synthesis problem.

$$\Longrightarrow i = k_2 v^2 \tag{219}$$

At this point we should look at figure 3 to see the equivalent circuit.

An Analysis Problem

Next we shall present an analysis example. Let us say that we are given an input current of

$$i(t) = u(t) \tag{220}$$

$$i(t) = cos(wt) \tag{221}$$

$$I(z_0) = 1 \tag{222}$$

$$V(z_0) = Z(z_0) \tag{223}$$

$$Z(z_0) = \frac{1}{z_0 + k_1 + k_2 \coprod^{2-1}} \tag{224}$$

This last fraction actually can be obtained using any of the following symbolic programming languages: 1) Macsyma; 2) Reduce; 3) PL1; and 4) Lisp.

In reference 2 this particular result tabulated, but we must change our variable

$$z_0 = x_0^{-1} \tag{225}$$

$$V(x_0^{-1} + k_1) = 1 - k_2 V \coprod^{2-1} \tag{226}$$

$$V(1 + k_1 x_0) = x_0 - k_2 x_0 V \coprod \tag{227}$$

$$V = (1 + k_1 x_0)^{-1} x_0 - k_2 (1 + k_1 x_0)^{-1} x_0 (V \coprod V) \tag{228}$$

The inverse transformations can be obtained by partial fractions and the time domain response of the system follows as

$$v(t) = \frac{1}{k_1}(1 - e^{-k_1 t}) - \frac{k_2}{k_1^3}(1 - 2k_1 t e^{-k_1 t} - e^{-2k_1 t}) + \ldots \tag{229}$$

When the input current is given as

$$i(t) = cos(wt) \tag{230}$$

similar steps will be followed and we have

$$I(z_0) = \frac{z_0^2}{z_0^2 + w^2} \tag{231}$$

$$z_0 = x_0^{-1} \tag{232}$$

$$V = (1 + k_1 x_0)^{-1}(1 + w^2 x_0^2)^{-1} x_0 - k_2 (1 + k_1 x_0)^{-1} x_0 [V \coprod V] \tag{233}$$

In table IV of reference 2, this generating power series is obtained using a symbolic program. Inverse transformation by partial fractions yields finally

$$v(t) = \frac{1}{k_1^2 + w^2}[-k_1 e^{-k_1 t} + k_1 cos(wt) + w sin(wt)] + \ldots \tag{234}$$

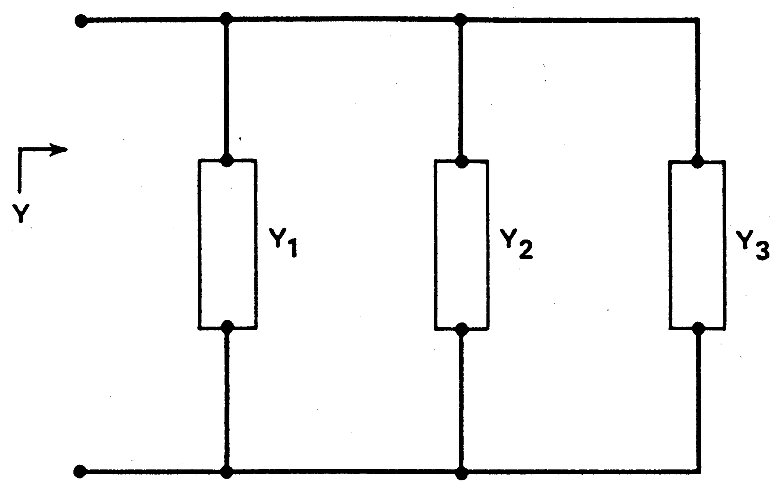

Figure 3.- An analysis problem.

Fréchet Differential of the Response

Generating power series or equivalently Laplace-Borel transforms of the responses of nonlinear systems assume the existence of analyticity throughout the regime. However, we are fully aware of the fact that the loss of analyticity is very important and as explained in reference 4 is equivalent to the loss of Fréchet differentiability of the response, and hence, to the bifurcations of the response. Bifurcations in between regimes do take place at the critical values of the system's parameters and we can account for them by monitoring the Fréchet differential of the response.

To fix ideas, let us consider the nonlinear circuit problem :

$$\frac{d}{dt}v + k_1 v + k_2 v^2 = i(t) \tag{235}$$

where $i(t)$ represents the input current. Let us denote the Fréchet differential of the response by $\delta N[v,\eta]$ which is given by its definition as

$$\delta N[v,\eta] = \lim_{\epsilon \longrightarrow 0}\left(\frac{v[i+\epsilon\eta] - v[i]}{\epsilon}\right) \tag{236}$$

or in Laplace-Borel transform domain,

$$\delta\mathcal{N}[V,\Omega] = \lim_{\epsilon \longrightarrow 0}\left(\frac{V[i+\epsilon\eta] - V[i]}{\epsilon}\right). \tag{237}$$

where V, Ω are the Laplace-Borel transforms of v, η, respectively. From the main theorem, we have

$$V(i) = G(z_0, \coprod).[\mathcal{LB}(i) + v(0)] \tag{238}$$

Similarly, using the main theorem once more we can write the output transformation for an input $i + \epsilon\eta$ as

$$V(i+\epsilon\eta) = G(z_0, \coprod).[\mathcal{LB}(i+\epsilon\eta) + v(0)] \tag{239}$$

since $\mathcal{LB}\ (i+\epsilon\eta) = \mathcal{LB}\ (i) + \epsilon\ \mathcal{LB}(\eta)$ we have

$$\delta\mathcal{N}[V,\Omega] = \lim_{\epsilon \longrightarrow 0} G(z_0, \coprod)\mathcal{LB}(\eta) \tag{240}$$

If $\Omega(z_0) = \mathcal{LB}(\eta)$ then

$$\delta\mathcal{N}(V,\Omega) = \lim_{\epsilon \longrightarrow 0} G(z_0, \coprod).\Omega(z_0) \tag{241}$$

or if we take $\eta(t) = u(t)$ (i.e., unit step function), then $\Omega(z_0) = 1$ and the Fréchet differential becomes

$$\delta\mathcal{N}[V,\Omega] = G(z_0, \coprod) \times 1 \tag{242}$$

or

$$\delta\mathcal{N}[V,\Omega] = G(z_0, \coprod) \tag{243}$$

which states that the Fréchet differential in Laplace-Borel transform domain is given by the transfer function of the system.

It is quite straightforward to generalize the previous result as

$$X(f) = G(z_0, \coprod).\left[F(z_0) + \sum_{i=1}^{n} \alpha_i z_0^{i-1} \frac{d^{i-1}}{dx^{i-1}} x(0)\right] \tag{244}$$

and

$$X(f + \epsilon\eta) = G(z_0, \coprod) \cdot \left[F(z_0) + \epsilon\Omega(z_0) + \sum_{i=1}^{n} \alpha_i z_0^{i-1} \frac{d^{i-1}}{dx^{i-1}} x(0) \right] \tag{245}$$

and hence

$$\delta\mathcal{N}[F, \Omega] = G(z_0, \coprod) \times \Omega(z_0) \tag{246}$$

or with $\eta(t) = u(t)$; i.e., unit step function

$$\delta\mathcal{N}[F, \Omega] = G(z_0, \coprod) \tag{247}$$

Hence, we have the following theorem

Theorem 10 (Fréchet Differential) *: The Fréchet differential of the response of a nonlinear dynamical system with polynomial type of nonlinearities is given in terms of the system's tranfer function and the variable of the Laplace-Borel transform as :*

$$\delta\mathcal{N}(X) = G(z_0, \coprod) \tag{248}$$

CONCLUDING REMARKS

We have demonstrated that the general response of nonlinear dynamical systems can be expressed in terms of their transfer functions in an analogous way to the linear systems.

We defined the transfer functions as the generalized series for the response of the nonlinear dynamical system which is initially at rest and which is loaded by a unit step function.

These transfer functions are obtainable through symbolic computer algebra and currently we have one of the following languages available to us: 1) Macsyma, 2) Reduce, and 3) PL1.

Analyticity of the response is assumed when the total response is expressed in terms of the transfer functions.

As a result of the last theorem on the Fréchet differential the loss of analyticity implies the loss of Fréchet differential (ref. 4), and the loss of Fréchet differential implies the loss of transfer function (i.e., at Bifurcation points) we can not determine the transfer functions.

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16. Abstract

An analytical functional can be expressed as a sum of some nonlinear functional expansions which we shall call Fliess's generalized expansions. These nonlinear functional expansions are analogous to Fourier series or integral expansions of response functions of linear systems. The shuffle product which is the characteristic of the noncommutative algebra introduced plays a very significant role in this approach. Moreover what makes this approach more attractive is the possibility of doing all of the noncommutative algebra on a computer in any of the currently available symbolic programming languages such as Macsyma, Reduce, PL1, and Lisp.

Nonlinear functional expansions for the solution of nonlinear ordinary differential equations can be summarized by the newly introduced Laplace-Borel transforms. Some properties of these transforms are obtained by the second author earlier. Some further properties will be given in this paper for the first time.

The main theorem of the paper gives the transform of the response of the nonlinear system as a Cauchy product of its transfer function which is introduced for the first time here and the transform of the input function of the system together with memory effects.

Applications of this new transfer-function approach are given using nonlinear electronic circuits. Two categories of applications are presented, namely,

- analysis of circuits
- synthesis of circuits.

We would like to remind the reader that various other examples can be given from other nonlinear dynamical systems; for example nonlinear aerodynamics, nonlinear flight mechanics in which cases these two classes of problems can be called either direct problems or inverse problems.

17. Key Words (Suggested by Author(s))
Transfer function
Linear systems
Fourier-Borel transforms

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